ANSWERS

Problem 1.1 -- A : triangular load version --

Question a)

This can be obtained straightforwardly by following the same steps as for a simple bar element. The only thing new is the non-constant q load, but that comes down to a one-line modification of the standard Maple script for bar elements. The final expressions are:

$$\begin{aligned} &-Hz'' = \frac{qx}{\ell} \\ &x = 0 \rightarrow z = z_1, V = -V_1 \quad \Rightarrow \quad K^{(e)} = \frac{H}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \quad f^{(e)} = \frac{q\ell}{6}\begin{bmatrix} 1 \\ 2 \end{bmatrix}; \\ &x = \ell \rightarrow z = z_2, V = V_2 \end{aligned}$$

Question b)

The approach would be the same, but the ODE would change because of the distributed load. The solution becomes much more complex in this case, and expressing K in parametric form becomes impractical. Moreover, linear shape functions would not be enough to derive f through the equivalent work principle anymore. This can be seen from the change from a parabola to a catenary shape for the cable. In fact, no polynomial shape functions of any degree would lead to an exact solution in this case.

Problem 1.2

Question a)

Due to the support conditions and arrangement of elements, only vertical DOFs are possible in this problem. Elements (2) and (3) deform with a combination of shear and cable (P-system). Using the definition of Problem 1.1 and the well-known extension element (lecture notes), we can list the DOFs, matrices and vectors as:

$$element\ 1\ (cable): \qquad DOFs\ w_1, w_2, K^{(1)} = \frac{H}{2\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, f^{(1)} = \begin{bmatrix} \frac{2q\ell}{6} \\ \frac{2q\ell}{3} \end{bmatrix}$$

$$element\ 2\ (shear + cable): \quad DOFs\ w_2, w_3$$

$$element\ 3\ (shear + cable): \quad DOFs\ w_3, w_4$$

$$element\ 4\ (extension): \qquad DOFs\ w_4, w_5, K^{(4)} = \frac{EA}{2\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, f^{(4)} = \begin{bmatrix} \frac{2p\ell}{2} \\ \frac{2p\ell}{2} \end{bmatrix}$$

Question b)

K for the shear problem has been done in class and is similar to the procedure in Question 1. Adding the cable can be done directly by recognizing the two ODEs are precisely compatible, and just summing up the stiffnesses is enough. Solving the P-system ODE from scratch will also give the same result. Important here is to motivate the choice for a P-system, which in this case is straightforward since the question already states the shear beam and cable are perfectly bonded and therefore deform together. Following the same procedure as in Problem 1.1, we arrive at:

$$element\ 2\ (shear+cable):\quad DOFs\ w_2, w_3, K^{(2)} = \frac{k+H}{\ell}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}, f^{(2)} = \begin{bmatrix}\frac{q\ell}{2}\\ \frac{q\ell}{2}\end{bmatrix}$$

$$element\ 3\ (shear+cable):\quad DOFs\ w_3, w_4, K^{(3)} = \frac{k+H}{\ell}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

Question c)

Use the element matrices obtained previously to assemble the final 5x5 stiffness matrix and the final force vector including contributions from the triangular load on the cable, the constant load on the p-system and the constant distributed load on the extension bar. With all of this in mind, we get:

$$\begin{bmatrix}125 & -125 & 0 & 0 & 0\\ -125 & 125+750 & -750 & 0 & 0\\ 0 & -750 & 750+750 & -750 & 0\\ 0 & 0 & -750 & 750+750 & -500\\ 0 & 0 & 0 & -500 & 50\end{bmatrix}\begin{bmatrix}w_1\\ w_2\\ w_3\\ w_4\\ w_5\end{bmatrix} = \begin{bmatrix}4\\ 8+6\\ 6\\ 10\\ 10\end{bmatrix}$$

The constraints here can be applied simply by striking rows and columns related to DOFs at fixed nodes. This leads to a reduced system:

$$\begin{bmatrix}875 & -750 & 0\\ -750 & 1500 & -750\\ 0 & -750 & 1250\end{bmatrix}\begin{bmatrix}w_2\\ w_3\\ w_4\end{bmatrix} = \begin{bmatrix}14\\ 6\\ 10\end{bmatrix}$$

solving :

$w_2 = 0.0665$ m; $w_3 = 0.0589$ m; $w_4 = 0.0434$ m;

Question d)

This question involves some basic postprocessing. Go back to the same ODE used for Problem 1.1, apply the DOF values we just obtained and solve it for the displacement at midspan. In other words, solve the ODE:

$$-Hw'' = \frac{qx}{2\ell}$$

with BC:

$x = 0;\quad w = 0;$

$x = 2\ell;\quad w = w_2$

and simply compute $w(\ell) = 0.04525$ m

Question e)

Yes, overlapping elements sharing the same nodes would work here. This can be seen from the nature of the assembling and from the nature of the ODE that describes this P-system. Care must however be taken if there are element loads, as it would make no sense to account for them twice. The same approach would not work for a combination of E-B bending and cable. The difference in the order of the ODEs being combined would not allow for a simple summation like this: the displacement fields coming from the two parts would not be compatible everywhere.

Problem 1.1 **-- B : sinusoidal load version --**

Question a)

This can be obtained straightforwardly by following the same steps as for a simple bar element. The only thing new is the non-constant q load, but that comes down to a one-line modification of the standard Maple script for bar elements. The final expressions are:

$$\begin{aligned}&-Hz'' = q_0 \sin\left(\frac{\pi x}{\ell}\right)\\ &x = 0 \rightarrow z = z_1, V = -V_1 \quad \Rightarrow \quad K^{(e)} = \frac{H}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \quad f^{(e)} = \frac{q\ell}{\pi}\begin{bmatrix} 1 \\ 1 \end{bmatrix};\\ &x = \ell \rightarrow z = z_2, V = V_2\end{aligned}$$

Question b)

The approach would be the same, but the ODE would change because of the distributed load. The solution becomes much more complex in this case, and expressing K in parametric form becomes impractical. Moreover, linear shape functions would not be enough to derive f through the equivalent work principle anymore. This can be seen from the change from a parabola to a catenary shape for the cable. In fact, no polynomial shape functions of any degree would lead to an exact solution in this case.

Problem 1.2

Question a)

Due to the support conditions and arrangement of elements, only vertical DOFs are possible in this problem. Elements (2) and (3) deform with a combination of shear and cable (P-system). Using the definition of Problem 1.1 and the well-known extension element (lecture notes), we can list the DOFs, matrices and vectors as:

$$\textit{element}\ 1\ (\textit{cable}): \qquad \textit{DOFs}\ w_1, w_2, K^{(1)} = \frac{H}{2\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, f^{(1)} = \begin{bmatrix} \frac{2q\ell}{\pi} \\ \frac{2q\ell}{\pi} \end{bmatrix}$$

$$\textit{element}\ 2\ (\textit{shear} + \textit{cable}): \quad \textit{DOFs}\ w_2, w_3$$

$$\textit{element}\ 3\ (\textit{shear} + \textit{cable}): \quad \textit{DOFs}\ w_3, w_4$$

$$\textit{element}\ 4\ (\textit{extension}): \qquad \textit{DOFs}\ w_4, w_5, K^{(4)} = \frac{EA}{2\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, f^{(4)} = \begin{bmatrix} \frac{2p\ell}{2} \\ \frac{2p\ell}{2} \end{bmatrix}$$

Question b)

K for the shear problem has been done in class and is similar to the procedure in Question 1. Adding the cable can be done directly by recognizing the two ODEs are precisely compatible, and just summing up the stiffnesses is enough. Solving the P-system ODE from scratch will also give the same result. Important here is to motivate the choice for a P-system, which in this case is straightforward since the question already states the shear beam and cable are perfectly bonded and therefore deform together. Following the same procedure as in Problem 1.1, we arrive at:

$$element\ 2\ (shear + cable): \quad DOFs\ w_2, w_3, K^{(2)} = \frac{k+H}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$element\ 3\ (shear + cable): \quad DOFs\ w_3, w_4, K^{(3)} = \frac{k+H}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, f^{(3)} = \begin{bmatrix} \frac{q\ell}{2} \\ \frac{q\ell}{2} \end{bmatrix}$$

Question c)

Use the element matrices obtained previously to assemble the final 5x5 stiffness matrix and the final force vector including contributions from the triangular load on the cable, the constant load on the p-system and the constant distributed load on the extension bar. With all of this in mind, we get:

$$\begin{bmatrix} 500 & -500 & 0 & 0 & 0 \\ -500 & 500+750 & -750 & 0 & 0 \\ 0 & -750 & 750+750 & -750 & 0 \\ 0 & 0 & -750 & 750+125 & -125 \\ 0 & 0 & 0 & -125 & 125 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 6 \\ 6+\frac{24}{\pi} \\ \frac{24}{\pi} \end{bmatrix}$$

The constraints here can be applied simply by striking rows and columns related to DOFs at fixed nodes. This leads to a reduced system:

$$\begin{bmatrix} 1250 & -750 & 0 \\ -750 & 1500 & -750 \\ 0 & -750 & 875 \end{bmatrix}\begin{bmatrix} w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ 6+\frac{24}{\pi} \end{bmatrix}$$

solving :

$w_2 = 0.0429$ m; $w_3 = 0.0582$ m; $w_4 = 0.0655$ m;

Question d)

This question involves some basic postprocessing. Go back to the ODE for the shear element, apply the DOF values we just obtained and solve it for the shear force at midspan. In other words, solve the ODE:

$$-kw'' = q_0$$

with BC:

$$x = 0; \quad w = w_3;$$

$$x = \ell; \quad w = w_4$$

and simply compute $V(\ell/2) = 3.65$ kN

Question e)

Yes, overlapping elements sharing the same nodes would work here. This can be seen from the nature of the assembling and from the nature of the ODE that describes this P-system. Care must however be taken if there are element loads, as it would make no sense to account for them twice. The same approach would not work for a combination of E-B bending and cable. The difference in the order of the ODEs being combined would not allow for a simple summation like this: the displacement fields coming from the two parts would not be compatible everywhere.

Problem 1.1 **-- C : trapezoidal load version --**

Question a)

This can be obtained straightforwardly by following the same steps as for a simple bar element. The only thing new is the non-constant *q* load, but that comes down to a one-line modification of the standard Maple script for bar elements. The final expressions are:

$$\begin{aligned} &-Hz'' = q_1 + \frac{(q_2-q_1)x}{\ell} \\ &x=0 \rightarrow z=z_1, V=-V_1 \\ &x=\ell \rightarrow z=z_2, V=V_2 \end{aligned} \quad \Rightarrow \quad K^{(e)} = \frac{H}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \quad f^{(e)} = \begin{bmatrix} \frac{q_1\ell}{3}+\frac{q_2\ell}{6} \\ \frac{q_1\ell}{6}+\frac{q_2\ell}{3} \end{bmatrix};$$

Question b)

The approach would be the same, but the ODE would change because of the distributed load. The solution becomes much more complex in this case, and expressing *K* in parametric form becomes impractical. Moreover, linear shape functions would not be enough to derive *f* through the equivalent work principle anymore. This can be seen from the change from a parabola to a catenary shape for the cable. In fact, no polynomial shape functions of any degree would lead to an exact solution in this case.

Problem 1.2

Question a)

Due to the support conditions and arrangement of elements, only vertical DOFs are possible in this problem. Elements (2) and (3) deform with a combination of shear and cable (P-system). Using the definition of Problem 1.1 and the well-known extension element (lecture notes), we can list the DOFs, matrices and vectors as:

$$element\ 1\ (cable): \qquad DOFs\ w_1, w_2, K^{(1)} = \frac{H}{2\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, f^{(1)} = \begin{bmatrix} \frac{(2q_1+q_2)\ell}{3} \\ \frac{(2q_2+q_1)\ell}{3} \end{bmatrix}$$

$$element\ 2\ (shear+cable): \quad DOFs\ w_2, w_3$$

$$element\ 3\ (shear+cable): \quad DOFs\ w_3, w_4$$

$$element\ 4\ (extension): \qquad DOFs\ w_4, w_5, K^{(4)} = \frac{EA}{2\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, f^{(4)} = \begin{bmatrix} \frac{2p\ell}{2} \\ \frac{2p\ell}{2} \end{bmatrix}$$

Question b)

K for the shear problem has been done in class and is similar to the procedure in Question 1. Adding the cable can be done directly by recognizing the two ODEs are precisely compatible, and just summing up the stiffnesses is enough. Solving the P-system ODE from scratch will also give the same result. Important here is to motivate the choice for a P-system, which in this case is straightforward since the question already states the shear beam and cable are perfectly bonded and therefore deform together. Following the same procedure as in Problem 1.1, we arrive at:

$$element\ 2\ (shear + cable): \quad DOFs\ w_2, w_3, K^{(2)} = \frac{k+H}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, f^{(2)} = \begin{bmatrix} \frac{q\ell}{2} \\ \frac{q\ell}{2} \end{bmatrix}$$

$$element\ 3\ (shear + cable): \quad DOFs\ w_3, w_4, K^{(3)} = \frac{k+H}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Question c)

Use the element matrices obtained previously to assemble the final 5x5 stiffness matrix and the final force vector including contributions from the triangular load on the cable, the constant load on the p-system and the constant distributed load on the extension bar. With all of this in mind, we get:

$$\begin{bmatrix} 125 & -125 & 0 & 0 & 0 \\ -125 & 125+750 & -750 & 0 & 0 \\ 0 & -750 & 750+750 & -750 & 0 \\ 0 & 0 & -750 & 750+750 & -500 \\ 0 & 0 & 0 & -500 & 50 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{bmatrix} = \begin{bmatrix} 20/3 \\ 28/3+6 \\ 6 \\ 10 \\ 10 \end{bmatrix}$$

The constraints here can be applied simply by striking rows and columns related to DOFs at fixed nodes. This leads to a reduced system:

$$\begin{bmatrix} 875 & -750 & 0 \\ -750 & 1500 & -750 \\ 0 & -750 & 1250 \end{bmatrix}\begin{bmatrix} w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 46/3 \\ 6 \\ 10 \end{bmatrix}$$

solving :

$w_2 = 0.0705$ m; $w_3 = 0.0618$ m; $w_4 = 0.0451$ m;

Question d)

This question involves some basic postprocessing. Go back to the same ODE used for Problem 1.1, apply the DOF values we just obtained and solve it for the displacement at midspan. In other words, solve the ODE:

$$-Hw'' = \frac{qx}{2\ell}$$

with BC:

$x = 0;\quad w = 0;$

$x = 2\ell;\quad w = w_2$

and simply compute $w(\ell) = 0.05125$ m

Question e)

Yes, overlapping elements sharing the same nodes would work here. This can be seen from the nature of the assembling and from the nature of the ODE that describes this P-system. Care must however be taken if there are element loads, as it would make no sense to account for them twice. The same approach would not work for a combination of E-B bending and cable. The difference in the order of the ODEs being combined would not allow for a simple summation like this: the displacement fields coming from the two parts would not be compatible everywhere.

Problem 2 **-- A : static determinate version --**

a) This so-called Timoshenko beam is described in terms of shear and bending stiffness. For a rectangular cross section we cannot use the full cross sectional area so that is why we use the term effective shear area. The resulting stiffnesses can then be found with:

$$GA_{eff} = G\tfrac{5}{6}bh = \frac{E}{2(1+\upsilon)}\frac{5bh}{6} = \frac{5Ebh}{12(1+\nu)} \quad [\text{N}]$$

$$EI \;= E\tfrac{1}{12}bh^3 \quad [\text{Nmm}^2]$$

b) The statement is incomplete. The ratio between the two stiffnesses is dimensional so this is not a proper ratio to consider. For a Poisson ratio of 0.25 this ratio e.g., yields:

$$\frac{GA_{eff}}{EI} = \frac{\dfrac{5Ebh}{12(1+\nu)}}{\dfrac{Ebh^3}{12}} = \frac{5}{(1+\nu)h^2} = \frac{4}{h^2} \quad [1/\text{mm}^2]$$

c) In case of a static determinate system the reduced second order ODE can be used. Explain how you find this. The moment distribution as a function is required, demonstrate how you find this expression. In this case only two BC are required to solve two integration constants. In case the standard Timoshenko approach is used clearly express the kinematic, constitutive and equilibrium conditions used to find these ODE's. Also show all four BC needed to solve the four integration constants.

d) For loadcase A, the distributed load, the deflection at midspan becomes:

due to shear

$$w_{mid} = \frac{ql^2}{8GA_{eff}} + \frac{5ql^4}{384EI} = \left(1 + \frac{ql^2}{8GA_{eff}} \times \frac{384EI}{5ql^4}\right)\frac{5ql^4}{384EI} = \left(1 + \frac{48}{25}\frac{h^2(1+\nu)}{l^2}\right)\frac{5ql^4}{384EI}$$

For loadcase B, the concentrated force, the deflection at midspan becomes:

$$w_{mid} = \frac{Fl}{4GA_{eff}} + \frac{Fl^3}{48EI} = \left(1 + \frac{Fl}{4GA_{eff}}\frac{48EI}{Fl^3}\right)\frac{Fl^3}{48EI} = \left(1 + \frac{60}{25}\frac{h^2(1+\nu)}{l^2}\right)\frac{Fl^3}{48EI}$$

These answers could be found with basic formulas or by using a MAPLE script based on c).

e) With a maximum of 5% deflection at midspan due to shear, we can find:

$q - load:$

$$\frac{48}{25}\frac{h^2(1+\nu)}{l^2} \overset{req}{<} \frac{1}{20} \Leftrightarrow \frac{h^2}{l^2} < \frac{25}{960(1+\nu)}$$

$$\frac{h}{l} < \sqrt{\frac{25}{960(1+\nu)}} \quad with\ \nu = 0,25\ results\ in:$$

$$\frac{h}{l} < \sqrt{\frac{25}{960(1+\frac{1}{4})}} = \frac{h}{l} < \sqrt{\frac{25}{1200}} = 0,1443$$

$$\frac{h}{l} < 0,1443 \quad or: \quad \frac{l}{h} > 6,92$$

$F - load:$

$$\frac{60}{25}\frac{h^2(1+\nu)}{l^2} \overset{req}{<} \frac{1}{20} \Leftrightarrow \frac{h^2}{l^2} < \frac{25}{1200(1+\nu)}$$

$$\frac{h}{l} < \sqrt{\frac{25}{1200(1+\nu)}} \quad with\ \nu = 0,25\ results\ in:$$

$$\frac{h}{l} < \sqrt{\frac{25}{1200(1+\frac{1}{4})}} = \frac{h}{l} < \sqrt{\frac{25}{1500}} = 0,1291$$

$$\frac{h}{l} < 0,1291 \quad or: \quad \frac{l}{h} > 7,75$$

f) In case the static system changes, e.g., turns from static determinate into static indeterminate, the deflection due to shear will remain the same but the part due to bending will reduce. The influence of the deformation due to shear will then increase. Therefor the criterion must be adjusted accordingly. In that case the slenderness criterion *l/h* must increase.

Problem 2 **-- B : static indeterminate version --**

a) This so-called Timoshenko beam is described in terms of shear and bending stiffness. For a rectangular cross section we cannot use the full cross sectional area so that is why we use the term effective shear area. The resulting stiffnesses can then be found with:

$$GA_{eff} = G\tfrac{5}{6}bh = \frac{E}{2(1+\upsilon)}\frac{5bh}{6} = \frac{5Ebh}{12(1+\nu)} \quad [\text{N}]$$

$$EI \quad = E\tfrac{1}{12}bh^3 \quad [\text{Nmm}^2]$$

b) The statement is incomplete. The ratio between the two stiffnesses is dimensional so this is not a proper ratio to consider. For a Poisson ratio of 0.25 this ratio e.g., yields:

$$\frac{GA_{eff}}{EI} = \frac{\dfrac{5Ebh}{12(1+\nu)}}{\dfrac{Ebh^3}{12}} = \frac{5}{(1+\nu)h^2} = \frac{4}{h^2} \quad [1/\text{mm}^2]$$

c) Since this is a static indeterminate beam the Timoshenko approach with two ODE's should be solved. Explain how you find this. Clearly express the kinematic, constitutive and equilibrium conditions used to find these ODE's. Also show all four BC needed to solve the four integration constants.

d) For loadcase A, the distributed load, the deflection at midspan becomes:

due to shear

$$w_{mid} = \frac{ql^2}{8GA_{eff}} + \frac{2ql^4}{384EI} = \left(1 + \frac{ql^2}{8GA_{eff}} \times \frac{384EI}{2ql^4}\right)\frac{2ql^4}{384EI} = \left(1 + \frac{48}{10}\frac{h^2(1+\nu)}{l^2}\right)\frac{2ql^4}{384EI}$$

For loadcase B, the concentrated force, the deflection at midspan becomes:

$$w_{mid} = \frac{Fl}{4GA_{eff}} + \frac{7Fl^3}{768EI} = \left(1 + \frac{Fl}{4GA_{eff}}\frac{768EI}{7Fl^3}\right)\frac{7Fl^3}{768EI} = \left(1 + \frac{192}{35}\frac{h^2(1+\nu)}{l^2}\right)\frac{7Fl^3}{768EI}$$

These answers could be found with basic formulas or by using a MAPLE script based on c).

e) With a maximum of 5% deflection at midspan due to shear, we can find:

$q - load:$

$$\frac{48}{10}\frac{h^2(1+\nu)}{l^2} \overset{req}{<} \frac{1}{20} \iff \frac{h^2}{l^2} < \frac{10}{960(1+\nu)}$$

$$\frac{h}{l} < \sqrt{\frac{10}{960(1+\nu)}} \quad with\ \nu = 0,25\ results\ in:$$

$$\frac{h}{l} < \sqrt{\frac{10}{960(1+\frac{1}{4})}} = \frac{h}{l} < \sqrt{\frac{1}{120}} = 0.09129$$

$$\frac{h}{l} < 0.09129 \quad or: \quad \frac{l}{h} > 10.95$$

$F - load:$

$$\frac{192}{35}\frac{h^2(1+\nu)}{l^2} \overset{req}{<} \frac{1}{20} \iff \frac{h^2}{l^2} < \frac{35}{3840(1+\nu)}$$

$$\frac{h}{l} < \sqrt{\frac{35}{3840(1+\nu)}} \quad with\ \nu = 0,25\ results\ in:$$

$$\frac{h}{l} < \sqrt{\frac{35}{3840(1+\frac{1}{4})}} = \frac{h}{l} < \sqrt{\frac{7}{960}} = 0.0854$$

$$\frac{h}{l} < 0.0854 \quad or: \quad \frac{l}{h} > 11.7$$

f) In case the static system changes, e.g. turns from static indeterminate into static determinate, the deflection due to shear will remain the same but the part due to bending will increase. The influence of the deformation due to shear will then reduce. Therefor the criterion must be adjusted accordingly. In that case the slenderness criterion *l/h* must be lowered.

Problem 2 **-- C : static indeterminate version --**

a) This so-called Timoshenko beam is described in terms of shear and bending stiffness. For a rectangular cross section we cannot use the full cross sectional area so that is why we use the term effective shear area. The resulting stiffnesses can then be found with:

$$GA_{eff} = G\tfrac{5}{6}bh = \frac{E}{2(1+\upsilon)}\frac{5bh}{6} = \frac{5Ebh}{12(1+\nu)} \quad [\text{N}]$$

$$EI \quad = E\tfrac{1}{12}bh^3 \quad [\text{Nmm}^2]$$

b) The statement is incomplete. The ratio between the two stiffnesses is dimensional so this is not a proper ratio to consider. For a Poisson ratio of 0.25 this ratio e.g., yields:

$$\frac{GA_{eff}}{EI} = \frac{\dfrac{5Ebh}{12(1+\nu)}}{\dfrac{Ebh^3}{12}} = \frac{5}{(1+\nu)h^2} = \frac{4}{h^2} \quad [1/\text{mm}^2]$$

c) Since this is a static indeterminate beam the Timoshenko approach with two ODE's should be solved. Explain how you find this. Clearly express the kinematic, constitutive and equilibrium conditions used to find these ODE's. Also show all four BC needed to solve the four integration constants.

d) For loadcase A, the distributed load, the deflection at midspan becomes:

due to shear

$$w_{mid} = \frac{ql^2}{8GA_{eff}} + \frac{ql^4}{384EI} = \left(1 + \frac{ql^2}{8GA_{eff}} \times \frac{384EI}{ql^4}\right)\frac{ql^4}{384EI} = \left(1 + \frac{48}{5}\frac{h^2(1+\nu)}{l^2}\right)\frac{ql^4}{384EI}$$

For loadcase B, the concentrated force, the deflection at midspan becomes:

$$w_{mid} = \frac{Fl}{4GA_{eff}} + \frac{Fl^3}{192EI} = \left(1 + \frac{Fl}{4GA_{eff}}\frac{192EI}{Fl^3}\right)\frac{Fl^3}{192EI} = \left(1 + \frac{48}{5}\frac{h^2(1+\nu)}{l^2}\right)\frac{Fl^3}{192EI}$$

These answers could be found with basic formulas or by using a MAPLE script based on c).

e) With a maximum of 5% deflection at midspan due to shear, we can find:

for both loads:

$$\frac{48h^2(1+\nu)}{5l^2} \overset{req}{<} \frac{1}{20} \quad \Leftrightarrow \quad \frac{h^2}{l^2} < \frac{5}{960(1+\nu)}$$

$$\frac{h}{l} < \sqrt{\frac{5}{960(1+\nu)}} \quad \textit{with } \nu = 0{,}25 \textit{ results in:}$$

$$\frac{h}{l} < \sqrt{\frac{5}{960(1+\frac{1}{4})}} = \frac{h}{l} < \sqrt{\frac{1}{240}} = 0.06454$$

$$\frac{h}{l} < 0.06454 \quad or: \quad \frac{l}{h} > 15.5$$

f) In case the static system changes, e.g. turns from static indeterminate into static determinate, the deflection due to shear will remain the same but the part due to bending will increase. The influence of the deformation due to shear will then reduce. Therefor the criterion must be adjusted accordingly. In that case the slenderness criterion *l/h* must be lowered.

Problem 3 **-- A : load left half span, compare with beam version --**

a) This curved beam is constrained such that it will act as an arch. Essential for this is the constrained horizontal displacements at the supports. Due to the geometry $z(x)$ of the arch the compressive force in the structure will have an internal lever arm which will result in a reduction of the bending moment compared to a standard straight beam. It will also reduce deflection considerable. Since the arch is also fully clamped, both the force distribution and the deflection $w(x)$ can only be found with the so-called differential equations for arches:

$$EIw'''' = q - Hz''$$

This problem must be solved with MAPLE. The load can be modelled with a *Heaviside* function to use one field for the entire span. (two fields are of course also fine)

$$q(x) = 10 \times (1 - Heaviside(x - l/2)) \qquad \text{[kN/m]} \qquad \text{(only left half of the span)}$$

The geometry was given as:

$$z(x) = \frac{-300x^2(l-x)^2}{10^7} \qquad \text{[m]}$$

Homogeneous boundary conditions used are:

$$w(0) = 0; \quad \varphi(0) = 0; \quad w(l) = 0; \quad \varphi(l) = 0;$$

Solving the ODE with these boundary conditions will result in a solution for the displacement field $w(x)$ in which the horizontal component H of the axial compression in the arch is still unknown. The additional condition to solve H is:

$$-\frac{\mathrm{d}z}{\mathrm{d}x}\frac{\mathrm{d}w}{\mathrm{d}x} = 0 \quad \text{(no axial deformation)}$$

Once the displacement field is solved all related distributions can be found, such as:

$$\varphi = -w'; \quad \kappa = -w''; \quad M = EI\kappa; \quad V_b = M'; \quad V_a = -Hz'$$

V is the vertical component of the force in the arch and can be split in a part due to bending and a part due to the compressive force.

b) Solving this problem results in a horizontal component of the compressive force in the arch H = 694.44 kN. The support reactions can be found as:

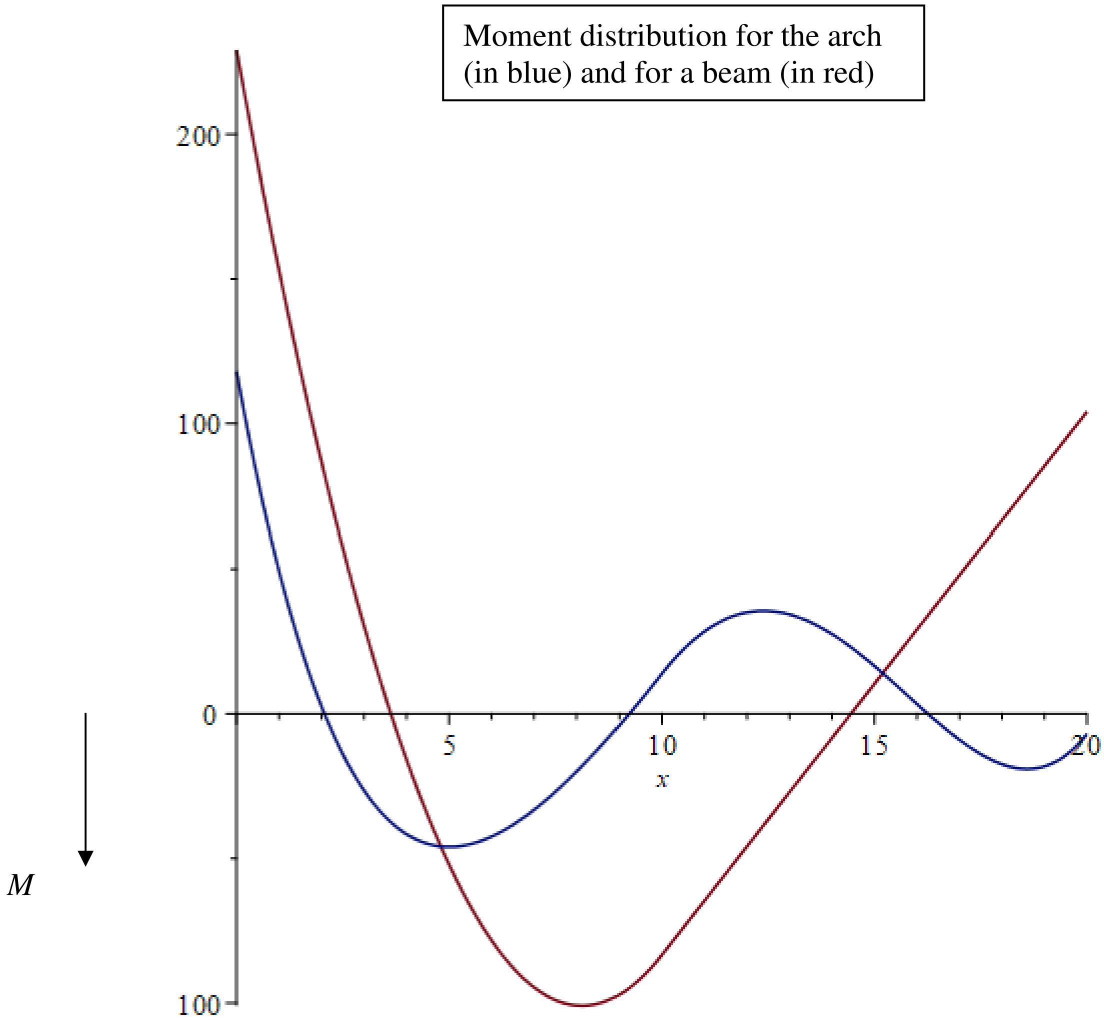
$$A_H = 694.44 \text{ kN } (\rightarrow); \quad A_V = V_b(0) + V_a(0) = V_b(0) = 81.25 \text{ kN } (\uparrow)$$
$$A_T = -M(0) = 118{,}06 \text{ kNm (anti clockwise)} \quad \text{mind the minus!}$$
$$B_H = 694.44 \text{ kN } (\uparrow); \quad B_V = -V_b(l) - V_a(l) = -V_b(l) = 18.75 \text{ kN } (\uparrow) \quad \text{mind the minus!}$$
$$B_T = +M(l) = 6.94 \text{ kNm (anti clockwise)}$$

NOTE:
1) Due to the special function of the geometry, the arch reaches the support horizontally since z'=0. So only the vertical component of the force distribution due to bending is contributing to the support reaction at the supports.
2) Sum of vertical forces is 100 kN which is identical to the vertical load.

c) The moment distribution M in [kNm] can be found with MAPLE. Check for the correct signs or deformation symbols and add the numbers.

At the clamped supports the moments are:

$$M(0) = -118.06 \quad \text{kNm}; \quad M(l) = 6.94 \quad \text{kNm};$$



d) Mention in this answer that not only the force distribution in terms of moment distribution is relevant but also the stiffness and thus the required cross sectional dimensions (weight) to satisfy deformation requirements e.g. maximum sag of approx. 1/300 of the span. With the given stiffness the beam is a no go. Further items to explain are the vertical resulting force in the structure (is identical so of no concern) and the additional axial component which will contribute to normal stresses as well. Grading depends on argumentation used.

Problem 3 **-- B : load right half span, compare with beam version --**

a) This curved beam is constrained such that it will act as an arch. Essential for this is the constrained horizontal displacements at the supports. Due to the geometry $z(x)$ of the arch the compressive force in the structure will have an internal lever arm which will result in a reduction of the bending moment compared to a standard straight beam. It will also reduce deflection considerable. Since the arch is also fully clamped, both the force distribution and the deflection $w(x)$ can only be found with the so-called differential equations for arches:

$$EIw'''' = q - Hz''$$

This problem must be solved with MAPLE. The load can be modelled with a *Heaviside* function to use one field for the entire span. (two fields are of course also fine)

$$q(x) = 10 \times (Heaviside(x - l/2)) \quad \text{[kN/m]} \qquad \text{(only right half of the span)}$$

The geometry was given as:

$$z(x) = \frac{-300x^2(l-x)^2}{10^7} \qquad \text{[m]}$$

Homogeneous boundary conditions used are:

$$w(0) = 0; \quad \varphi(0) = 0; \quad w(l) = 0; \quad \varphi(l) = 0;$$

Solving the ODE with these boundary conditions will result in a solution for the displacement field $w(x)$ in which the horizontal component H of the axial compression in the arch is still unknown. The additional condition to solve H is:

$$-\frac{dz}{dx}\frac{dw}{dx} = 0 \quad \text{(no axial deformation)}$$

Once the displacement field is solved all related distributions can be found, such as:

$$\varphi = -w'; \quad \kappa = -w''; \quad M = EI\kappa; \quad V_b = M'; \quad V_a = -Hz'$$

V is the vertical component of the force in the arch and can be split in a part due to bending and a part due to the compressive force.

b) Solving this problem results in a horizontal component of the compressive force in the arch H = 694.44 kN. The support reactions can be found as:

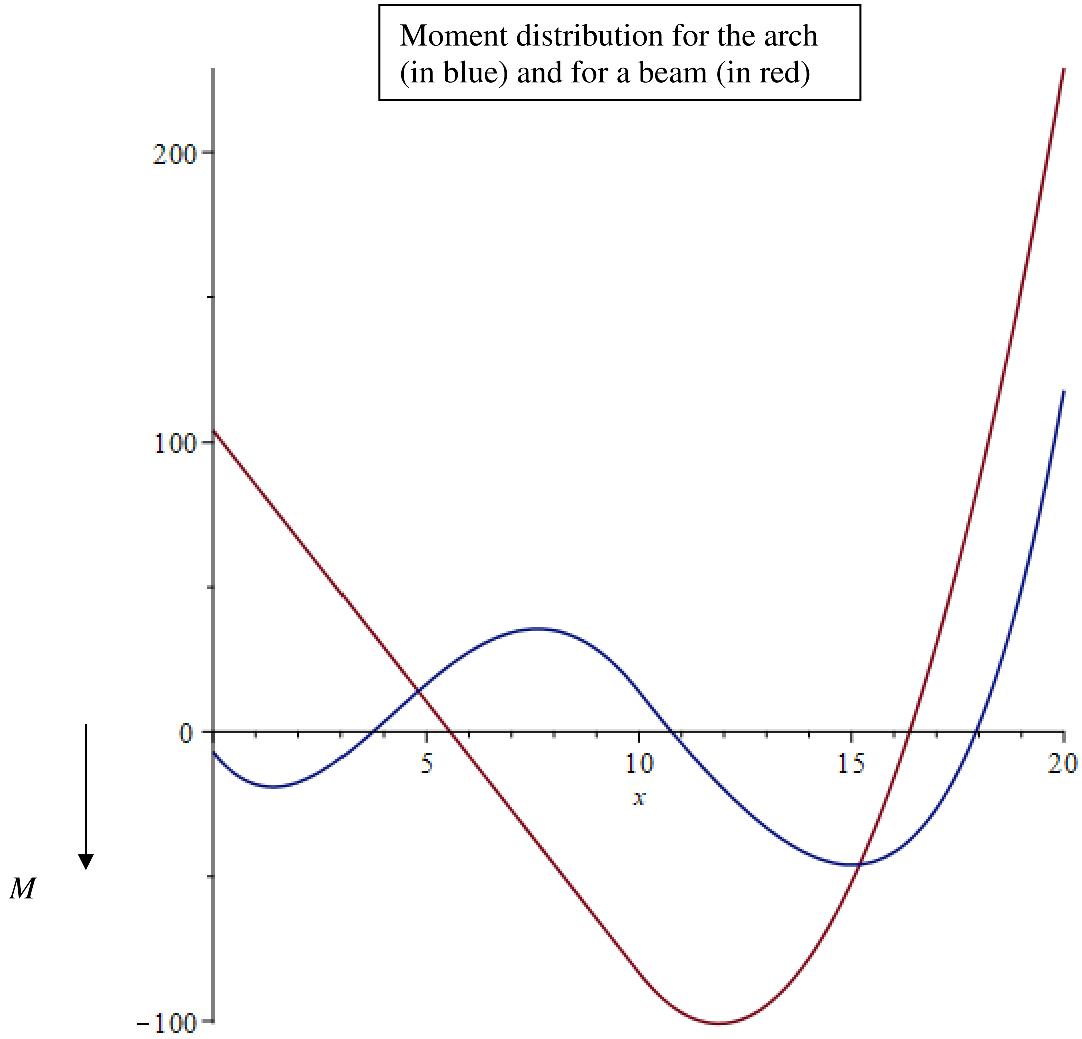
$$A_H = 694.44 \quad \text{kN } (\uparrow); \qquad A_V = V_b(0) + V_a(0) = V_b(0) = 18.75 \quad \text{kN } (\uparrow)$$
$$A_T = -M(l) = 6.94 \qquad \text{kNm (clockwise)} \quad \text{mind the minus!}$$
$$B_H = 694.44 \quad \text{kN } (\rightarrow); \quad B_V = -V_b(l) - V_a(l) = -V_b(l) = 81.25 \quad \text{kN } (\uparrow) \quad \text{mind the minus!}$$
$$B_T = M(l) = -118{,}06 \quad \text{kNm (clockwise)}$$

NOTE: 1) Due to the special function of the geometry, the arch reaches the support horizontally since z'=0. So only the vertical component of the force distribution due to bending is contributing to the support reaction at the supports.
2) Sum of vertical forces is 100 kN which is identical to the vertical load.

c) The moment distribution M in [kNm] can be found with MAPLE. Check for the correct signs or deformation symbols and add the numbers.

At the clamped supports the moments are:

$$M(0) = 6.94 \quad \text{kNm}; \quad M(l) = -118.06 \quad \text{kNm};$$



d) Mention in this answer that not only the force distribution in terms of moment distribution is relevant but also the stiffness and thus the required cross sectional dimensions (weight) to satisfy deformation requirements e.g. maximum sag of approx. 1/300 of the span. With the given stiffness the beam is a no go. Further items to explain are the vertical resulting force in the structure (is identical so of no concern) and the additional axial component which will contribute to normal stresses as well. Grading depends on argumentation used.

Problem 3 **-- C : load left half span, compare with parabolic arch version --**

a) This curved beam is constrained such that it will act as an arch. Essential for this is the constrained horizontal displacements at the supports. Due to the geometry $z(x)$ of the arch the compressive force in the structure will have an internal lever arm which will result in a reduction of the bending moment compared to a standard straight beam. It will also reduce deflection considerable. Since the arch is also fully clamped, both the force distribution and the deflection $w(x)$ can only be found with the so-called differential equations for arches:

$$EIw'''' = q - Hz''$$

This problem must be solved with MAPLE. The load can be modelled with a *Heaviside* function to use one field for the entire span. (two fields are of course also fine)

$$q(x) = 10 \times (1 - Heaviside(x - l/2)) \qquad \text{[kN/m]} \qquad \text{(only left half of the span)}$$

The geometry was given as:

$$z(x) = \frac{-300x^2(l-x)^2}{10^7} \qquad \text{[m]}$$

Homogeneous boundary conditions used are:

$$w(0) = 0; \quad \varphi(0) = 0; \quad w(l) = 0; \quad \varphi(l) = 0;$$

Solving the ODE with these boundary conditions will result in a solution for the displacement field $w(x)$ in which the horizontal component H of the axial compression in the arch is still unknown. The additional condition to solve H is:

$$-\frac{dz}{dx}\frac{dw}{dx} = 0 \quad \text{(no axial deformation)}$$

Once the displacement field is solved all related distributions can be found, such as:

$$\varphi = -w'; \quad \kappa = -w''; \quad M = EI\kappa; \quad V_b = M'; \quad V_a = -Hz'$$

V is the vertical component of the force in the arch and can be split in a part due to bending and a part due to the compressive force.

b) Solving this problem results in a horizontal component of the compressive force in the arch H = 694.44 kN. The support reactions can be found as:

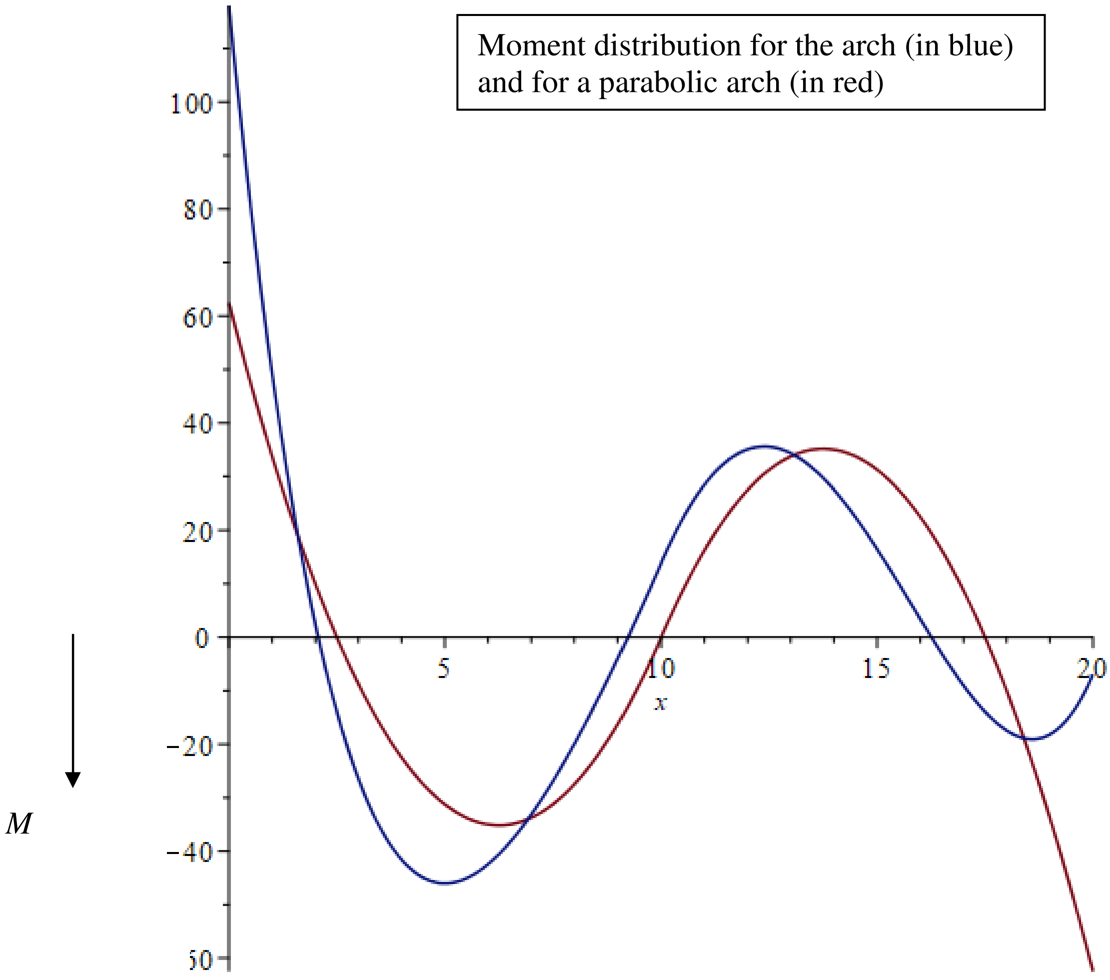
$$A_H = 694.44 \quad \text{kN } (\rightarrow); \quad A_V = V_b(0) + V_a(0) = V_b(0) = 81.25 \quad \text{kN } (\uparrow)$$
$$A_T = -M(0) = 118{,}06 \quad \text{kNm (anti clockwise)} \quad \text{mind the minus!}$$
$$B_H = 694.44 \quad \text{kN } (\uparrow); \quad B_V = -V_b(l) - V_a(l) = -V_b(l) = 18.75 \quad \text{kN } (\uparrow) \quad \text{mind the minus!}$$
$$B_T = +M(l) = 6.94 \quad \text{kNm (anti clockwise)}$$

NOTE: 1) Due to the special function of the geometry, the arch reaches the support horizontally since z'=0. So only the vertical component of the force distribution due to bending is contributing to the support reaction at the supports.
2) Sum of vertical forces is 100 kN which is identical to the vertical load.

c) The moment distribution M in [kNm] can be found with MAPLE. Check for the correct signs or deformation symbols and add the numbers.

At the clamped supports the moments are:

$$M(0) = -118.06 \quad \text{kNm}; \quad M(l) = 6.94 \quad \text{kNm};$$



d) Mention in this answer that not only the force distribution in terms of moment distribution is relevant but also the stiffness and thus the required cross sectional dimensions (weight) to satisfy deformation requirements e.g. maximum sag of approx. 1/300 of the span. With the given stiffness a beam is a no go and difference in force distribution between the given design or a parabolic shape are primarily found at the supports. Further items to explain are the vertical resulting force in the structure (is in total identical so of no concern) and the difference in axial component which will contribute to normal stresses as well. Grading depends on argumentation used.